

GEOMETRIZATION OF CONTINUOUS CHARACTERS OF \mathbb{Z}_p^\times

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ABSTRACT. We define the p -adic trace of certain rank-one local systems on the multiplicative group over p -adic numbers, using Sekiguchi and Suwa's unification of Kummer and Artin-Schrier-Witt theories. Our main observation is that, for every non-negative integer n , the p -adic trace defines an isomorphism of abelian groups between local systems whose order divides $(p-1)p^n$ and ℓ -adic characters of the multiplicative group of p -adic integers of depth less than or equal to n .

Motivation. Let p and ℓ be distinct primes and let q be a power of p . Let G be a connected algebraic group over \mathbb{F}_q . To geometrize a character $\psi : G(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ one pushes forward the Lang central extension

$$0 \rightarrow G(\mathbb{F}_q) \rightarrow G \xrightarrow{\text{Lang}} G \rightarrow 0, \quad \text{Lang}(x) = \text{Fr}(x) - x,$$

by ψ^{-1} and obtains a local system \mathcal{L}_ψ on G . The trace of Frobenius of \mathcal{L}_ψ equals ψ ; which is to say that \mathcal{L}_ψ and ψ correspond under the functions-sheaves dictionary. Thus, we think of \mathcal{L}_ψ as *the geometrization of ψ* . Let $\mathcal{C}(G)$ be the abelian group (under tensor product) consisting of \mathcal{L}_ψ as ψ ranges over $\text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times)$; in other words, $\mathcal{C}(G)$ is the group of irreducible summands of $\text{Lang}_! \overline{\mathbb{Q}}_\ell$. Trace of Frobenius defines an isomorphism of abelian groups

$$(1) \quad t_{\text{Fr}} : \mathcal{C}(G) \xrightarrow{\sim} \text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times);$$

see [1, Sommes Trig.] and [3, Ex. 1.1.3].

In this note we obtain an analogue of the above isomorphism for \mathbb{G}_m over p -adic numbers.

Theorem. *The work of Sekiguchi and Suwa provides an isomorphism between the abelian group of rank-one local systems on $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$ whose order divides $(p-1)p^n$ and the abelian group of characters of \mathbb{Z}_p^\times of depth less than or equal to n , for every non-negative integer n .*

The rest of this note concerns the proof of this theorem.

Unification of Kummer with Artin-Schrier-Witt. Henceforth, we assume that p is an *odd* prime. Fix a non-negative integer n and a primitive p^n -th root of unity $\zeta \in \overline{\mathbb{Q}}_p$. Set $R = \mathbb{Z}_p[\zeta]$, $K = \mathbb{Q}_p(\zeta)$. The main theorem of Sekiguchi and Suwa on the unification of Kummer and Artin-Schreier-Witt theories provides us with:

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- an exact sequence

$$0 \rightarrow \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathcal{Y} \xrightarrow{f} \mathcal{X} \rightarrow 0$$

of commutative group schemes over R ,

- isomorphisms $\mathcal{Y}_K := \mathcal{Y} \otimes_R K \xrightarrow{\cong} \mathbb{G}_{m,K}^{n+1}$ and $\mathcal{X}_K \rightarrow \mathbb{G}_{m,K}^{n+1}$,
- isomorphisms $\mathcal{Y}_{\mathbb{F}_p} \xrightarrow{\cong} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$ and $\mathcal{X}_{\mathbb{F}_p} \xrightarrow{\cong} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$,

such that the following diagram commutes

$$\begin{array}{ccccccc} \mathbb{G}_{m,K} & \xleftarrow{m} & \mathbb{G}_{m,K}^{n+1} & \xleftarrow{\quad} & \mathcal{Y}_K & \longrightarrow & \mathcal{Y} & \xleftarrow{\quad} & \mathcal{Y}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \\ \downarrow \theta & & \downarrow \gamma & & \downarrow f_K & & \downarrow f & & \downarrow f_{\mathbb{F}_p} & & \downarrow \text{Lang} \\ \mathbb{G}_{m,K} & \xleftarrow{\alpha} & \mathbb{G}_{m,K}^{n+1} & \xleftarrow{\quad} & \mathcal{X}_K & \longrightarrow & \mathcal{X} & \xleftarrow{\quad} & \mathcal{X}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \end{array}$$

Here, $\theta(x) = x^{(p-1)p^n}$, m denotes the multiplication map, γ and α are defined by

$$\gamma(x_0, \dots, x_n) = (x_0^{p-1}, \frac{x_1^p}{x_2}, \frac{x_2^p}{x_3}, \dots, \frac{x_n^p}{x_{n-1}}), \quad \alpha(x_0, x_1, \dots, x_n) := \frac{(x_0 x_1 x_2 x_3 \cdots x_n)^{p^n}}{x_1 x_2^p x_3^{p^2} \cdots x_n^{p^{n-1}}},$$

and f_K and $f_{\mathbb{F}_p}$ are the restrictions of f to the generic and special fibre, respectively. The main theorem of Sekiguchi and Suwa result was announced in [5]. A preprint containing a proof appeared subsequently [6]. According to Sekiguchi, the main tools of this preprint have been published in [7]. For a general overview see [8].

The p -adic trace function. Let $\mathbf{K}(\mathbb{G}_{m,K})$ denote the group (under tensor product) of local systems that are irreducible summands of $\theta_! \overline{\mathbb{Q}}_\ell$. One can easily check that all the squares in the above diagram are Cartesian; moreover, it is clear that all the vertical arrows are Galois covers of order $(p-1)p^n$. It follows that the diagram above determines a canonical isomorphism of groups

$$(2) \quad S : \mathbf{K}(\mathbb{G}_{m,K}) \xrightarrow{\cong} \mathbf{C}(\mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}).$$

We define the p -adic trace function by

$$(3) \quad \begin{aligned} \mathfrak{T}\mathfrak{r}_n : \mathbf{K}(\mathbb{G}_{m,K}) &\longrightarrow \text{Hom}(\mathbb{G}_m(\mathbb{F}_p) \times \mathbb{W}_n(\mathbb{F}_p), \overline{\mathbb{Q}}_\ell^\times) \\ \mathcal{K} &\mapsto t_{\text{Fr}}(S(\mathcal{K})). \end{aligned}$$

It follows at once from (1) and (2) that $\mathfrak{T}\mathfrak{r}_n$ is a canonical isomorphism.

Relationship to continuous characters of \mathbb{Z}_p^\times . Since p is odd, the exponential map defines an isomorphism of algebraic \mathbb{F}_p -groups

$$(4) \quad \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \xrightarrow{\cong} \mathbb{W}_{n+1,\mathbb{F}_p}^*$$

where $\mathbb{W}_{n+1,\mathbb{F}_p}^*$ refers to the group scheme of units in the Witt ring scheme $\mathbb{W}_{n+1,\mathbb{F}_p}$ (see [2]) and therefore an isomorphism

$$(5) \quad \mathbb{G}_m(\mathbb{F}_p) \times \mathbb{W}_n(\mathbb{F}_p) = \mathbb{Z}/(p-1) \times \mathbb{Z}/p^n \xrightarrow{\cong} \mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p).$$

Accordingly, we can think of the p -adic trace as a character of $\mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p)$. Composing with the quotient $\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p)$, we see that the p -adic trace can be interpreted as a continuous ℓ -adic character of \mathbb{Z}_p^\times .

Conversely, for every continuous character $\chi : \mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, there is a non-negative integer n such that $\chi(\mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p)) = \{1\}$. The smallest such n is known as

the depth of χ . We propose to think of $\mathcal{K}_\chi := \mathfrak{T}\mathfrak{r}_n^{-1}(\chi)$ as *the geometrization of χ* , when $\chi : \mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ is a continuous character of depth n . We do not discuss how to vary n in the present text.

We note that choosing an isomorphism of the form (5) is unappetizing, to quote Deligne. We hope, in time, to give a construction which does not depend on this choice.

Relationship to character sheaves. A character sheaf of $\mathbb{G}_{m, \overline{\mathbb{Q}_p}}$ is a perverse sheaf on $\mathbb{G}_{m, \overline{\mathbb{Q}_p}}$ (cohomologically) concentrated in degree 1 where it is a rank-one local system (see [4, §2]). Local systems on $\mathbb{G}_{m, \overline{\mathbb{Q}_p}}$ of order dividing $(p-1)p^n$ are precisely those that have a $\mathbb{Q}_p(\mu_{p^\infty})$ -rational structure; that is, they can be defined on $\mathbb{G}_{m, \mathbb{Q}_p(\mu_{p^\infty})}$. In this language, the main result of this note is the following: *the p -adic trace of every $\mathbb{Q}_p(\mu_{p^\infty})$ -rational character sheaf on $\mathbb{G}_{m, \overline{\mathbb{Q}_p}}$ is a continuous character $\mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ and, moreover, every continuous ℓ -adic character of \mathbb{Z}_p^\times is obtained in this manner, each one from a unique character sheaf of $\mathbb{G}_{m, \overline{\mathbb{Q}_p}}$.*

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REFERENCES

- [1] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie 1963-64 SGA 4 $\frac{1}{2}$. Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.
- [2] M. Greenberg, *Unit Witt Vectors*, Proc. Amer. Math. Soc. **13** (1962), 72–73.
- [3] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 131–210.
- [4] George Lusztig, *Character sheaves. I*, Adv. in Math. **56** (1985), no. 3, 193–237.
- [5] Tsutomu Sekiguchi and Noriyuki Suwa, *Théorie de Kummer-Artin-Schreier et applications*, J. Théor. Nombres Bordeaux **7** (1995), no. 1, 177–189. Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993).
- [6] ———, *On the unified Kummer-Artin-Schreier-Witt theory*, Math. Pures de Bordeaux C.N.R.S., Prepublication **11** (1999), no. 1, 1–94.
- [7] ———, *A note on extensions of algebraic and formal groups. V*, Japan. J. Math. (N.S.) **29** (2003), no. 2, 221–284. MR2035540 (2004m:14098)
- [8] Kazuyoshi Tsuchiya, *On the descriptions of $\mathbb{Z}/p^n\mathbb{Z}$ -torsors by the Kummer-Artin-Schreier-Witt theory*, available at <http://home.t00.itscom.net/tsuchiya/Research/ResearchPDF/03CHUOMATH51.pdf>.
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